In the limiting special case when $\lambda(y)\equiv 1$, relations (2.26) and formula (5.6) give the well-known expression for the fundamental solution of the two-dimensional Laplace equation.

REFERENCES

1. BAKHVALOV N. S. and PANASENKO G. P., *Averaging of Processes in Periodic Media*. Nauka, Moscow, 1984.
2. POBEDRYA B. E., *Mechanics of Composite Materials*. Izd. Moskov. Gos. Univ., Moscow, 1984.
3. KALAMKAROV A. L., KUDRYAVTSEV B. A. and PARTON V. Z., The asymptotic averaging method in the mechanics of composites of regular structure. In *Itogi Nauki i Tekhniki. Ser. Mekhanika Deformiruemogo tverdogo tela* (VINITI, Moscow) **19**, 78–147, 1987.
4. SANCHEZ-PALENCIA E., Boundary layers and edge effects in composites. *Lecture Notes in Physics* **272**, 121–192. Springer, Berlin, 1987.
5. KALAMKAROV A. L., KUDRYAVTSEV B. A. and PARTON V. Z., The boundary-layer method in the mechanics of the fracture of composites of periodic structure. *Prikl. Mat. Mekh.* **54**, 2, 322–328, 1990.
6. PARTON V. Z., KALAMKAROV A. L. and BORISKOVSKII V. G., Investigation of local fields in the vicinity of macrocracks in a composite material of periodic structure. *Fiz.-Khim. Mekhanika Materialov* No. 1, 3–9, 1990.
7. UFLYAND Ya. S., On some new integral transforms and their applications to problems of mathematical physics. In *Problems of Mathematical Physics*, pp. 93–106. Nauka, Leningrad, 1976.

Translated by D.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 852–862, 1991
Printed in Great Britain.
0021-8928/91 $15.00+.00

THE EIGENMODE EXPANSION METHOD FOR OSCILLATIONS OF AN ELASTIC BODY WITH INTERNAL AND EXTERNAL FRICTION†

I. A. PASHKOV and I. YE. TROYANOVSKII

Moscow

(*Received* 30 *January* 1990)

A method is proposed for solving dynamical problems for a viscoelastic body (the Kelvin–Voigt model) in a massless viscous medium. Interaction with the external medium produces on the boundary of the body stresses proportional to the rate of displacement. The model of external friction is that used for modelling dynamical processes in elastic media filling an infinite domain [1, 2]. The implementation of numerical methods of solution requires an equivalent restatement of the problem in a finite domain, using external viscous friction to allow for the radiation of energy at infinity.

FROM THE mathematical point of view, the eigenvalue spectral problem in the presence of friction is not self-adjoint and the eigenfunctions are not orthogonal. For an elastic body with friction, the

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 972–981, 1991.

existence of a series expansion of the solution of the stationary and non-stationary problems in the eigenfunctions of the spectral problem is proved. The coefficients of the expansion are determined from explicit generalized orthogonality relationships, which are obtained by the approach previously proposed in [3].

The basic assumptions of the proposed method are first examined as they apply to a finite-dimensional model of a viscoelastic system.

1. A SYSTEM WITH A FINITE NUMBER OF DEGREES OF FREEDOM

In the general case with both external and internal viscous friction, small oscillations of an arbitrary mechanical system with n degrees of freedom are described by a matrix equation with initial conditions

$$\mathbf{A}\mathbf{u}^{\cdot\cdot} + \mathbf{B}\mathbf{u}^{\cdot} + \mathbf{C}\mathbf{u} = \mathbf{f}(t) \tag{1.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}^{\cdot}(0) = \mathbf{v}_0 \tag{1.2}$$

where $\mathbf{u}$ and $\mathbf{f}$ are n-dimensional vectors of generalized coordinates and generalized forces, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $n \times n$ real symmetric matrices and $\mathbf{A}$ and $\mathbf{C}$ are non-singular.

In the stationary problem, the vector function $\mathbf{f}$ varies harmonically as $\mathbf{f} = \mathbf{F}e^{i\omega t}$ with a given frequency ω and amplitude $\mathbf{F}(\omega)$. Initial conditions are not imposed and instead the solution is required to be periodic with the same frequency ω: $\mathbf{u} = \mathbf{U}e^{i\omega t}$. As a result, we obtain a system of linear algebraic equations for the complex components of the required vector $\mathbf{U}$:

$$(-\omega^2\mathbf{A} + i\omega\mathbf{B} + \mathbf{C})\mathbf{U} = \mathbf{F} \tag{1.3}$$

With the stationary problem (1.3) we associate the spectral problem with the parameter λ:

$$(\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C})\mathbf{y} = \mathbf{0} \tag{1.4}$$

The solutions of problem (1.4) are the eigenvalues λ and the eigenvectors $\mathbf{y}(\mathbf{y} \neq \mathbf{0})$ of the quadratic pencil of operators in the finite-dimensional complex vector space C^n. Unlike the standard eigenvalue problem [4], Eq. (1.4) is non-linear in the spectral parameter λ. The eigenvalues are therefore complex and the eigenvectors do not necessarily form a basis.

We will linearize problem (1.4) by the spectral parameter, transferring to coordinate–velocity space of double dimensions. We will denote by $\mathbf{v}$ the vector $\lambda\mathbf{y}$ and rewrite Eq. (1.4) in the form of an equivalent system of two equations

$$\mathbf{C}\mathbf{y} + \lambda(\mathbf{B}\mathbf{y} + \mathbf{A}\mathbf{v}) = \mathbf{0}, \quad \mathbf{A}(\mathbf{v} - \lambda\mathbf{y}) = \mathbf{0} \tag{1.5}$$

or in matrix form

$$(\mathbf{P} + \lambda\mathbf{R})\mathbf{q} = \mathbf{0}, \quad \mathbf{P} = \begin{Vmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A} \end{Vmatrix}, \quad \mathbf{R} = \begin{Vmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{Vmatrix} \tag{1.6}$$

where $\mathbf{q} = \{\mathbf{y}, \mathbf{v}\}$ is an element in the space C^{2n}.

Note that the matrices $\mathbf{P}$ and $\mathbf{R}$ are symmetric, non-singular, and indefinite.

The non-homogeneous problem (1.3) similarly can be rewritten in the form

$$(\mathbf{P} + i\omega\mathbf{R})\mathbf{Q} = \mathbf{G} \tag{1.7}$$

where $\mathbf{Q} = \{\mathbf{U}, \mathbf{V}\}$ is the required vector of coordinate–velocity amplitudes and $\mathbf{G} = \{\mathbf{F}, \mathbf{0}\}$ is the element in the space C^{2n} formed from the vector $\mathbf{F}$ and the null vector $\mathbf{0}$.

In the space of vector functions, the Cauchy problem (1.1) and (1.2) for the second-order equation in the generalized coordinate–velocity vector $\mathbf{x} = \{\mathbf{u}, \mathbf{v}\}$ is

$$\mathbf{P}\mathbf{x} + \mathbf{R}\mathbf{x}^{\cdot} = \mathbf{g}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.8}$$

where the vector $\mathbf{x}_0$ is formed from the vectors $\mathbf{u}_0$ and $\mathbf{v}_0$, and the vector $\mathbf{g}$ is formed from the vector $\mathbf{f}$ and the null vector.

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ of the spectral problems (1.4) and (1.8) are the roots of the characteristic equation

$$\det(\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}) \equiv -(\det \mathbf{A})^{-1} \det(\mathbf{P} + \lambda\mathbf{R}) = 0 \tag{1.9}$$

Proposition 1.1. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be the solutions of problem (1.4) that correspond to different eigenvalues λ_1 and λ_2. Then we have two generalized orthogonality relationships:

$$(\mathbf{y}_1, \mathbf{y}_2)_1 \equiv \mathbf{y}_1 \cdot \mathbf{B}\mathbf{y}_2 + (\lambda_1 + \lambda_2)\, \mathbf{y}_1 \cdot \mathbf{A}\mathbf{y}_2 = 0 \tag{1.10}$$

$$(\mathbf{y}_1, \mathbf{y}_2)_2 \equiv \mathbf{y}_1 \cdot \mathbf{C}\mathbf{y}_2 - \lambda_1\lambda_2 \mathbf{y}_1 \cdot \mathbf{A}\mathbf{y}_2 = 0 \tag{1.11}$$

The centred dot denotes the convolution of two vectors.

Indeed, if $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of problem (1.4), then $\mathbf{q}_1 = \{y_1, \lambda_1 y_1\}$ and $\mathbf{q}_2 = \{y_2, \lambda_2 y_2\}$ satisfy the equations

$$(\mathbf{P} + \lambda_1\mathbf{R})\, \mathbf{q}_1 = \mathbf{0}, \quad (\mathbf{P} + \lambda_2\mathbf{R})\, \mathbf{q}_2 = \mathbf{0}$$

From the convolution of the first equationn with the vector $\mathbf{q}_2$ subtract the convolution of the second equation with the vector $\mathbf{q}_1$. Using the symmetry of the matrices $\mathbf{P}$ and $\mathbf{R}$, we obtain

$$(\lambda_1 - \lambda_2)\, \mathbf{q}_1 \cdot \mathbf{R}\mathbf{q}_2 = 0$$

Hence for $\lambda_1 \neq \lambda_2$ we have

$$\mathbf{q}_1 \cdot \mathbf{R}\mathbf{q}_2 = 0, \quad \mathbf{q}_1 \cdot \mathbf{P}\mathbf{q}_2 = 0 \tag{1.12}$$

Expanding (1.12), we obtain (1.10) and (1.11).

Proposition 1.2. Assume that the system of eigenvectors $\{\mathbf{q}_k\}$ of the spectral problem (6.1) forms a basis in the space C^{2n}. Then the scalar squares $(\mathbf{y}_k, \mathbf{y}_k)_1$ and $(\mathbf{y}_k, \mathbf{y}_k)_2$ are non-zero for any eigenvector $\mathbf{y}_k$ which corresponds to a simple eigenvalue λ_k.

Proof. Assume that there exists a vector $\mathbf{y}_k$ that corresponds to a simple eigenvalue such that $(\mathbf{y}_k, \mathbf{y}_k)_1 = \mathbf{q}_k \cdot \mathbf{R} q_k = 0$. Since the system $\{\mathbf{q}_k\}$ forms a basis, any vector $\mathbf{a} \in C^{2n}$ can be represented in sum form

$$\mathbf{a} = C_1\mathbf{q}_1 + \ldots + C_{2n}\mathbf{q}_{2n}$$

Multiplying this relationship by the matrix $\mathbf{R}$ and convolving with $\mathbf{q}_k$, we obtain by the orthogonality relationship (1.10)

$$\mathbf{q}_k \cdot \mathbf{R}\mathbf{a} = \mathbf{a} \cdot \mathbf{R}\mathbf{q}_k = 0$$

The latter is possible for any $\mathbf{a}$ only if $\mathbf{q}_k = 0$, which contradicts the condition.

Note that the conditions of Proposition 1.2 are satisfied if all the roots of Eq. (1.9) are simple. From Propositions 1.3 and 1.4 (see below) it follows that we can omit in Proposition 1.2 the condition that the system of eigenvectors forms a basis.

We will show that with multiple eigenvectors the relationships (1.12) remain valid for vectors from the root subspaces corresponding to different eigenvalues.

We introduce a Keldysh canonical system [5] of eigenvectors and associated vectors $\{\mathbf{y}_k^\circ, \mathbf{y}_k^1, \ldots, \mathbf{y}_k^p\}$ corresponding to the eigenvalues $\lambda_k (1 \leqslant k \leqslant 2n)$. The eigenvectors $\mathbf{y}_k^\circ$ whose multiplicity $p+1$ depends on k satisfy Eq. (1.4) for $\lambda = \lambda_k$ and the associated vectors $\mathbf{y}_k^i$ $(i = 1, \ldots, p)$ satisfy the equations

$$(\lambda_k^2\mathbf{A} + \lambda_k\mathbf{B} + \mathbf{C})\,\mathbf{y}_k^1 + (2\lambda_k\mathbf{A} + \mathbf{B})\,\mathbf{y}_k^\circ = \mathbf{0} \qquad (1.13)$$

$$(\lambda_k^2\mathbf{A} + \lambda_k\mathbf{B} + \mathbf{C})\,\mathbf{y}_k^i + (2\lambda_k\mathbf{A} + \mathbf{B})\,\mathbf{y}_k^{i-1} + \mathbf{A}\mathbf{y}_k^{i-2} = \mathbf{0}, \qquad i > 1 \qquad (1.14)$$

To each chain consisting of an eigenvector and the associated vectors $\{\mathbf{y}_k^\circ, \mathbf{y}_k^1, \ldots, \mathbf{y}_k^p\}$ there corresponds a derived chain $\{\lambda_k\mathbf{y}_k^\circ, \ldots, \lambda_k\mathbf{y}_k^p + \mathbf{y}_k^{p-1}\}$. Then the vectors $\mathbf{q}_k^\circ = \{\mathbf{y}_k^\circ, \lambda_k\mathbf{y}_k^\circ\}, \ldots, \mathbf{q}_k^p = \{\mathbf{y}_k^p, \lambda_k\mathbf{y}_k^p + \mathbf{y}_k^{p-1}\}$ form a chain consisting of an eigenvector and the associated vectors of the spectral problem (1.6) that corresponds to the same eigenvalue λ_k, i.e. they satisfy the equations

$$(\mathbf{P} + \lambda_k\mathbf{R})\,\mathbf{q}_k^\circ = \mathbf{0} \qquad (1.15)$$

$$(\mathbf{P} + \lambda_k\mathbf{R})\,\mathbf{q}_k^m + \mathbf{R}\mathbf{q}_k^{m-1} = \mathbf{0}, \qquad m = 1, \ldots, p \qquad (1.16)$$

Proposition 1.3. Let $\mathbf{q}_1^m$ and $\mathbf{q}_2^j$ be elements of chains of eigenvectors and associated vectors of problem (1.6) that correspond to different eigenvalues λ_1 and λ_2 and are of length p_1 and p_2 respectively. Then for any $0 \leqslant m \leqslant p_1$, $0 \leqslant j \leqslant p_2$, we have the generalized orthogonality relationships

$$\mathbf{q}_1^m \cdot \mathbf{R}\mathbf{q}_2^j = 0, \qquad \mathbf{q}_1^m \cdot \mathbf{P}\mathbf{q}_2^j = 0 \qquad (1.17)$$

Proof. The case $m = j = 0$ is proved in Proposition 1.1. Let $m = 0, j = 1$. From the convolution of Eq. (1.15) for $k = 1$ with the vector $\mathbf{q}_2^1$ subtract the convolution of Eq. (1.16) for $k = 2$ with the vector $\mathbf{q}_1^\circ$. Using the symmetry of the matrices **P** and **R**, we obtain

$$(\lambda_1 - \lambda_2)\,\mathbf{q}_1^\circ \cdot \mathbf{R}\mathbf{q}_2^1 + \mathbf{q}_1^\circ \cdot \mathbf{R}\mathbf{q}_2^\circ = 0$$

The second term vanishes for $\lambda_1 \neq \lambda_2$, and therefore

$$\mathbf{q}_1^\circ \cdot \mathbf{R}\mathbf{q}_2^1 = 0 \qquad (1.18)$$

Let $m = 1, j = 1$. Subtracting the convolution of Eq. (1.16) for $k = 2$ with the vector $\mathbf{q}_1^1$ from the convolution of Eq. (1.16) for $k = 1$ with the vector $\mathbf{q}_2^1$, we obtain

$$(\lambda_1 - \lambda_2)\,\mathbf{q}_1^1 \cdot \mathbf{R}\mathbf{q}_2^1 + \mathbf{q}_1^\circ \cdot \mathbf{R}\mathbf{q}_2^1 - \mathbf{q}_1^1 \cdot \mathbf{R}\mathbf{q}_2^\circ = 0 \qquad (1.19)$$

The last two terms in (1.19) vanish by (1.18), and therefore $\mathbf{q}_1^1 \cdot \mathbf{R}\mathbf{q}_2^1 = 0$.

For other values of m and j, the first relationship in (1.17) is proved similarly by induction. Convolving Eq. (1.16) for $k = 1$ with the vector $\mathbf{q}_2^j$, we obtain the second relationship in (1.17).

The generalized orthogonality relationships (1.17) can be obtained from the biorthogonality relationships constructed in [6] if we note that for real symmetric matrices **A**, **B** and **C** the elements of the chains $\mathbf{z}_k^m$ of the conjugate of the spectral problem (1.4) corresponding to the eigenvalue $\bar{\lambda}_k$ are related to the vectors $\mathbf{y}_k^m$ by the conjugation operation $\mathbf{z}_k^m = \bar{\mathbf{y}}_k^m$.

From relationships (1.15) and (1.16) it also follows that the elements of a chain consisting of an eigenvector and the associated vectors are always linearly independent. For each eigenvalue λ_k

there exists at least one eigenvector. We cn show [5] that the sum of multiplicities of the eigenvectors corresponding to the same eigenvalue λ_k is equal to the multiplicity of the root λ_k in Eq. (1.12). Hence we obtain the following proposition.

Proposition 1.4. The canonical system of eigenvectors and associated vectors $\{\mathbf{q}_k{}^m\}$ of the spectral problem (1.6) forms a basis in the space C^{2n}.

If the system $\{\mathbf{q}_k{}^m\}$ is a basis, then the solution of the non-homogeneous problem (1.7) is representable in sum form

$$\mathbf{Q} = \sum_{k,\, m} C_k{}^m \mathbf{q}_k{}^m$$

where the coefficients of the expansion $C_k{}^\circ$ of the eigenvectors corresponding to simple eigenvalues are determined explicitly from the generalized orthogonality relationships (1.17):

$$C_k{}^\circ = \frac{\mathbf{G}\cdot\mathbf{q}_k{}^\circ}{(i\omega-\lambda_k)\,\mathbf{q}_k{}^\circ\cdot\mathbf{R}\mathbf{q}_k{}^\circ} = \frac{\mathbf{F}\cdot\mathbf{y}_k}{(i\omega-\lambda_k)\,(\mathbf{y}_k,\mathbf{y}_k)_1} \tag{1.20}$$

For multiple eigenvalues, the coefficients $C_k{}^m$ can be obtained by solving the system of linear algebraic equations

$$\sum_{k,\, j} (\mathbf{q}_l{}^m\cdot\mathbf{P}\mathbf{q}_k{}^j + i\omega\mathbf{q}_l{}^m\cdot\mathbf{R}\mathbf{q}_k{}^j)\, C_k{}^j = \mathbf{G}\cdot\mathbf{q}_l{}^m \tag{1.21}$$

whose order is equal to the multiplicity of the eigenvalue and its matrix is non-singular for $\omega \neq -i\lambda_k$. The non-singularity condition is always satisfied in the presence of viscosity, when the eigenvalues are complex. In the expression (1.21) it is implied that the vectors $\mathbf{q}_l{}^m$, $\mathbf{q}_k{}^j$ correspond to the same eigenvalue but belong to different root subspaces for $l \neq k$.

Similarly, for simple eigenvalues, the Cauchy problem (1.8) can be reduced to a system of independent differential equations of first order with initial conditions

$$x_k{}^\cdot + \lambda_k x_k = \mathbf{f}\cdot\mathbf{y}_k/(\mathbf{y}_k,\mathbf{y}_k)_1, \qquad x_k(0) = \mathbf{q}_k\cdot\mathbf{R}\mathbf{x}_0/(\mathbf{y}_k,\mathbf{y}_k)_1$$

The solution of this system produces the required function $u(t)$ in sum form

$$\mathbf{u} = \sum_k x_k(t)\,\mathbf{y}_k$$

In the presence of multiple eigenvalues, we obtain systems of coupled equations for the coefficients x_k, but the order of these systems does not exceed the multiplicity of the corresponding eigenvalue.

2. SYSTEMS WITH DISTRIBUTED PARAMETERS

Consider a viscoelastic Voigt body which fills the domain Ω with the boundary Γ. The motion of the body is described by the equations [7]

$$\nabla\cdot\boldsymbol{\sigma} - \rho\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{f} = 0$$
$$\boldsymbol{\sigma} = \left(\mathbf{c}_0 + \mathbf{c}_1\frac{\partial}{\partial t}\right)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{u}), \qquad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \text{ in } \Omega \tag{2.1}$$

Here $\mathbf{u}$ is the displacement vector, $\mathbf{f}$ is the distributed volume load vector, ρ is the volume density, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{c}_0 = \{c^0_{ijkl}\}$ is the tensor of elastic moduli, $\mathbf{c}_1 = \{c^1_{ijkl}\}$ is the tensor of the coefficients of viscosity and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the Cauchy strain-tensor differential operator.

We will assume that the tensors $\mathbf{c}_s$ have the following symmetry properties:

$$c^s_{ijkl} = c^s_{jikl} = c^s_{klij}, \quad s = 0, 1$$

Zero displacement are specified on a part of the boundary $\Gamma_0 \subset \Gamma$:

$$\mathbf{u} = 0 \tag{2.2}$$

On the remaining part of the boundary Γ_1 we have contact conditions with the external viscoelastic medium

$$\mathbf{n}\cdot\left(\mathbf{c}_0 + \mathbf{c}_1 \frac{\partial}{\partial t}\right)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{u}) = \left(\mathbf{b}_0 + \mathbf{b}_1 \frac{\partial}{\partial t}\right)\cdot\mathbf{u} \tag{2.3}$$

where $\mathbf{n}$ is the outer normal vector to the surface Γ_1, $\mathbf{b}_0$ and $\mathbf{b}_1$ are symmetric tensors of second rank of the stiffness and coefficients of viscosity of the external medium.

Initially, at a time $t = 0$, we know the displacement field $\mathbf{u}_0$ and the velocity field $\mathbf{v}_0$ in Ω:

$$\mathbf{u}\,|_{t=0} = \mathbf{u}_0, \quad \partial\mathbf{u}/\partial t\,|_{t=0} = \mathbf{v}_0 \tag{2.4}$$

Relationships (2.1)–(2.4) define a mixed boundary-value problem.

In the stationary case, the load $\mathbf{f}$ is harmonic $\mathbf{f} = \mathbf{F}e^{i\omega t}$ with a given frequency ω, and the solution is also sought in harmonic form $\mathbf{u} = \mathbf{U}e^{i\omega t}$. The function $\mathbf{U}(\mathbf{r})$ thus satisfies the relationships

$$\begin{gathered} \nabla\cdot[(\mathbf{c}_0 + i\omega\mathbf{c}_1)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{U})] + \rho\omega^2\mathbf{U} + \mathbf{F} = \mathbf{0} \text{ in } \Omega \\ \mathbf{U} = \mathbf{0} \text{ on } \Gamma_0, \quad \mathbf{n}\cdot(\mathbf{c}_0 + i\omega\mathbf{c}_1)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{U}) = (\mathbf{b}_0 + i\omega\mathbf{b}_1)\cdot\mathbf{U} \text{ on } \Gamma_1 \end{gathered} \tag{2.5}$$

With the non-homogeneous boundary-value problem (2.5) we associate a homogeneous spectral problem with the parameter λ:

$$\begin{gathered} \nabla\cdot[(\mathbf{c}_0 + \lambda\mathbf{c}_1)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{y})] - \rho\lambda^2\mathbf{y} = \mathbf{0} \text{ in } \Omega \\ \mathbf{y} = 0 \text{ on } \Gamma_0, \quad \mathbf{n}\cdot(\mathbf{c}_0 + \lambda\mathbf{c}_1)\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{y}) = (\mathbf{b}_0 + \lambda\mathbf{b}_1)\cdot\mathbf{y} \text{ on } \Gamma_1 \end{gathered} \tag{2.6}$$

Unlike the standard eigenvalue problem of elasticity theory [8], this problem is non-linear in the parameter λ (both the equation and the last boundary condition).

Problem (2.6) can also be restated as an eigenvalue problem for a quadratic pencil of unbounded operators.

Denote by $W_2{}^2(\Omega)$ the Sobolev space of functions that are square integrable on Ω^2 together with their first and second derivatives.

In the Hilbert space $H = [L_2(\Omega)]^3 \times [L_2(\Gamma_1)]^3$ with the scalar product defined as the sum of the scalar products in $L_2(\Omega)$ and $L_2(\Gamma_1)$, consider the operators P_0, P_1, P_2 defined on the elements y^+ of the lineal $D \subset H$:

$$D = \{y^+ \colon y^+ = (\mathbf{y}, \text{trace of } \mathbf{y} \text{ on } \Gamma_1),\ \mathbf{y} \in [W_2{}^2(\Omega)]^3,\ \mathbf{y} = 0 \text{ on } \Gamma_0\}$$

by the equalities

$$\begin{gathered} P_s y^+ = \{\nabla\cdot[\mathbf{c}_s\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{y})] \text{ in } \Omega,\ -\mathbf{n}\cdot\mathbf{c}_s\cdot\cdot\,\boldsymbol{\varepsilon}(\mathbf{y}) + \mathbf{b}_s\cdot\mathbf{y} \text{ on } \Gamma_1\} \\ P_2 y^+ = \{-\rho\mathbf{y}, \mathbf{0}\}, \quad s = 0, 1 \end{gathered} \tag{2.7}$$

Then the spectral problem (2.6) can be represented in the form

$$(\lambda^2 P_2 + \lambda P_1 + P_0)\, y^+ = 0 \tag{2.8}$$

Using Gauss' theorem and the symmetry properties of the tensors $\mathbf{c}_s$ and $\mathbf{b}_s$, we can show that the operators P_s for any elements y^+, $z^+ \in D$ satisfy the relationships

$$y^+ \cdot P_s z^+ = z^+ \cdot P_s y^+, \quad s = 0, 1, 2 \tag{2.9}$$

where the binary convolution operation between the elements y^+ and z^+ of the space H is defined by the formula

$$y^+ \cdot z^+ = \int_\Omega \mathbf{y} \cdot \mathbf{z}\, d\Omega + \int_{\Gamma_1} \mathbf{y} \cdot \mathbf{z}\, d\Gamma$$

For real tensors $\mathbf{c}_s$ and $\mathbf{b}_s$, relationships (2.9) imply that the operators P_s are symmetric.

Proposition 2.1. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions of the spectral problem (2.6) that correspond to different eigenvalues λ_1 and λ_2. Then we have the generalized orthogonality relationships

$$\begin{aligned}(\mathbf{y}_1, \mathbf{y}_2)_1 &\equiv \int_\Omega [\rho(\lambda_1 + \lambda_2)\, \mathbf{y}_1 \cdot \mathbf{y}_2 + \boldsymbol{\varepsilon}(\mathbf{y}_1) \cdot\cdot\, \mathbf{c}_1 \cdot\cdot\, \boldsymbol{\varepsilon}(\mathbf{y}_2)]\, d\Omega - \int_{\Gamma_1} \mathbf{y}_1 \cdot \mathbf{b}_1 \cdot \mathbf{y}_2\, d\Gamma = 0 \\ (\mathbf{y}_1, \mathbf{y}_2)_2 &\equiv \int_\Omega [\boldsymbol{\varepsilon}(\mathbf{y}_1) \cdot\cdot\, \mathbf{c}_0 \cdot\cdot\, \boldsymbol{\varepsilon}(\mathbf{y}_2) - \rho\lambda_1\lambda_2 \mathbf{y}_1 \cdot \mathbf{y}_2]\, d\Omega - \int_{\Gamma_1} \mathbf{y}_1 \cdot \mathbf{b}_0 \cdot \mathbf{y}_2\, d\Gamma = 0\end{aligned} \tag{2.10}$$

Proof. Following the logic of the proof of Proposition 1.1, we linearize Eq. (2.8) by the spectral parameter, i.e. we introduce the doubled space $H^2 = H \times H$ with the elements $q = \{q^\circ, q^1\}$, $q^\circ, q^1 \in H$ and define in this space the linear operators P and R with the definition domain $D^2 = D \times D$:

$$Pq = \{P_0 q^\circ - P_2 q^1\}, \quad Rq = \{P_1 q^\circ + P_2 q^1,\ P_2 q^\circ\} \tag{2.11}$$

Then Eq. (2.8) is equivalent to the equation

$$(P + \lambda R)\, q = 0 \tag{2.12}$$

The operators P and R are symmetric, because the operators P_s are symmetric. Therefore, defining in H^2 the binary convolution operation

$$q \cdot p = q^\circ \cdot p^\circ + q^1 \cdot p^1, \quad q, p \in H^2$$

we obtain

$$q_1 \cdot R q_2 = 0, \quad q_1 \cdot P q_2 = 0 \tag{2.13}$$

where $q_1 = \{y_1^+, \lambda_1 y_1^+\}$, $q_2 = \{y_2^+, \lambda_2 y_2^+\}$ are the solutions of Eq. (2.12) corresponding to the eigenvalues λ_1 and λ_2.

Expanding (2.13) by formulas (2.11) and (2.7) and applying Gauss' theorem, we obtain the relationships

$$(\mathbf{y}_1, \mathbf{y}_2)_1 = -q_1 \cdot R q_2 = 0, \quad (\mathbf{y}_1, \mathbf{y}_2)_2 = -q_1 \cdot P q_2 = 0$$

We can similarly generalize Proposition 1.3 to the case of a Hilbert space by introducing a canonical system of eigenelements and associated elements.

Unfortunately, the decision that the elements of a canonical system form a basis in a Hilbert space is not as simple as in Euclidean space.

Using the results of [6], we can only assert that under certain conditions on the operators P_s that ensure discreteness of the spectrum of the pencil (2.8) the canonical system of eigenelements and associated elements of the problem (2.13) is minimal, and the scalar squares $(\mathbf{y}, \mathbf{y})_1$, $(\mathbf{y}, \mathbf{y})_2$ of the eigenvectors of problem (2.6) are non-zero for simple eigenvalues.

Proposition 2.2. Assume that the solution of the boundary-value problem (2.5) admits of a double expansion in eigenfunctions and associated functions $\mathbf{y}_k^m$ of the problem (2.6):

$$\mathbf{U} = \sum_{k,m} C_k^m \mathbf{y}_k^m, \quad i\omega\mathbf{U} = \sum_{k,m} C_k^m (\lambda_k \mathbf{y}_k^m + \mathbf{y}_k^{m-1}) \quad (\mathbf{y}_k^{-1} \equiv 0) \tag{2.14}$$

Then the coefficients C_k^0 of the eigenfunctions $\mathbf{y}_k^0$ corresponding to simple eigenvalues λ_k are obtained from the formula

$$C_k^\circ = [(i\omega - \lambda_k)(\mathbf{y}_k^\circ, \mathbf{y}_k^\circ)_1]^{-1} \int_\Omega \mathbf{F}\cdot\mathbf{y}_k^\circ \, d\Omega \tag{2.15}$$

Relationship (2.15) is derived like (1.20) in Sec. 1. We can similarly repeat the entire argument concerning multiple eigenvalues and the solution of the non-stationary problem (2.1)–(2.4).

3. OSCILLATIONS OF AN IMMERSED ELASTIC CYLINDER UNDER THE ACTION OF INTERNAL PRESSURE

As an example, consider the problem of the oscillations of an elastic hollow cylinder immersed in a liquid and acted upon by periodic internal pressure.

In the cylindrical coordinate system r, φ, z, the elastic isotropic cylinder with Lamé constants λ and μ (possibly dependent on r) fills the volume $R_1 \leqslant r \leqslant R_2$, $0 \leqslant \varphi \leqslant 2\pi$, $-\infty < z < \infty$. The domain $r > R_2$ is occupied by an ideal liquid of density ρ_0 with velocity of sound c_0. At the interface between the liquid and the solid $r = R_2$ the radial stresses and displacement are assumed continuous. On the inner surface $r = R_1$, the harmonic pressure $p = p_0 e^{i\omega t}$ is given. When solving the stationary problem, we must also allow for the radiation conditions at infinity.

The relationships linking the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$ have the form

$$\begin{gathered} \boldsymbol{\sigma} = \lambda\nabla\cdot\mathbf{u}\mathbf{E} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \quad R_1 < r < R_2 \\ \boldsymbol{\sigma} = \rho_0 c_0^2 \nabla\cdot\mathbf{u}\mathbf{E}, \quad r > R_2 \end{gathered} \tag{3.1}$$

where $\mathbf{E}$ is the identity tensor of second rank.

Taking axial symmetry into account, we represent Eq. (3.1) in coordinate form:

$$\begin{gathered} \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} - \rho\frac{\partial^2 u_r}{\partial t^2} = 0 \\ \sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{u_r}{r} \equiv l u_r \\ \sigma_{\varphi\varphi} = (\lambda + 2\mu)\frac{u_r}{r} + \lambda\frac{\partial u_r}{\partial r} \end{gathered} \tag{3.2}$$

For $r > R_2$, we take $\lambda = \rho_0 c_0^2$, $\mu = 0$ in Eqs (3.2).

On the surface of the cylinder, we have the conditions

$$\sigma_{rr}|_{r=R_1} = -p_0 e^{i\omega t}$$
$$\sigma_{rr}|_{r=R_2-0} = \sigma_{rr}|_{r=R_2+0}, \quad u_r|_{r=R_2-0} = u_r|_{r=R_2+0} \tag{3.3}$$

Separating the variable $u_r = U(r)e^{i\omega t}$ and eliminating the stresses in (3.2), we obtain the equation of stationary oscillations with boundary conditions

$$LU + \rho\omega^2 U = 0, \quad LU \equiv \frac{d}{dr}\left[(\lambda + 2\mu)\frac{dU}{dr} + \lambda\frac{U}{r}\right] + \frac{2\mu}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right)$$
$$lU(R_1) = -p_0, \quad U(R_2 - 0) = U(R_2 + 0), \quad lU(R_2 - 0) = $$
$$= lU(R_2 + 0) \tag{3.4}$$

If we additionally require that the function U satisfies at infinity the Sommerfeld radiation conditions

$$\lim_{r\to\infty} \sqrt{r}\,|U| = \text{const}, \quad \lim_{r\to\infty} \sqrt{r}\left(\frac{dU}{dr} + i\frac{\omega}{c_0}U\right) = 0 \tag{3.5}$$

then the boundary value problems (3.4) and (3.5) are uniquely solvable [9].

For an acoustic medium, the function

$$U = B\frac{d}{dr}H_0^{(2)}\left(\frac{\omega r}{c_0}\right) \tag{3.6}$$

satisfies Eq. (3.4) for $r > R_2$ and the radiation conditions (3.5). In (3.6), B is an arbitrary constant and $H_0^{(2)}$ is the Hankel function of the second kind of zero order. Also

$$\sigma_{rr} = -\rho_0\omega^2 B H_0^{(2)}(\omega r/c_0)e^{i\omega t} \tag{3.7}$$

Then eliminating B from (3.6) and (3.7), we reduce the problem in the semi-infinite interval $R_1 \leqslant r < \infty$ to a problem in the finite interval $R_1 \leqslant r \leqslant R_3$ with $R_3 \geqslant R_2$, with the following boundary condition on the surface $r = R_3$:

$$lU = -\rho_0\omega^2 H_0^{(2)}(\omega r/c_0)\left[\frac{dH_0^{(2)}}{dr}\left(\frac{\omega r}{c_0}\right)\right]^{-1} U \tag{3.8}$$

Relationship (3.8) is meromorphic in the parameter ω. For the high-frequency range, we use the asymptotic representation of the Hankel function for large arguments,

$$H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\,e^{-i(z-\pi/4)}\left[1 + O\left(\frac{1}{z}\right)\right]$$

and, if necessary, increasing R_3, we replace condition (3.8) with the approximate relationship

$$lU(R_3) = -i\omega\rho_0 c_0 U(R_3) \tag{3.9}$$

which is linear in the parameter ω.

The validity of the passage from problems (3.4) and (3.8) to problems (3.4) and (3.9) depends on the robustness of the solution to small perturbations on the boundary. We will not consider this question here.

Formally, condition (3.9) in the original formulation is equivalent to the contact condition on the surface $r = R_3$ with viscous external medium: $\sigma_{rr} = -bdu_r\partial t$. The coefficient of viscosity b equals the wave resistance of the liquid $\rho_0 c_0$.

The problem with non-homogeneous boundary conditions (3.4) is always reducible to a problem with homogeneous boundary conditions by replacing the variable U with the variable V according to $U = V + U_0$, where the function U_0 satisfies the boundary conditions

$$lU_0(R_1) = -p_0, \quad U(R_2) = 0, \quad lU(R_2) = 0$$

Henceforth we will assume that the function U_0 is given. Then V is the solution of the following boundary-value problem:

$$\begin{gathered} LV + \rho\omega^2 V + f = 0, \quad f = LU_0 + \rho\omega^2 U_0 \\ (R_1 < r < R_3; \quad U_0 \equiv 0, \quad r > R_2) \\ lV(R_1) = 0, \quad V(R_2 - 0) = V(R_2 + 0) \\ lV(R_2 - 0) = lV(R_2 + 0), \quad lV(R_3) = -i\omega bV(R_3) \end{gathered} \tag{3.10}$$

With the stationary problem (3.10) we associate the spectral boundary-value problem with the parameter λ

$$\begin{gathered} Ly - \rho\lambda^2 y = 0, \quad R_1 < r < R_3 \\ ly(R_1) = 0, \quad y(R_2 - 0) = y(R_2 + 0) \\ ly(R_2 - 0) = ly(R_2 + 0), \quad ly(R_3) + \lambda by(R_3) = 0 \end{gathered} \tag{3.11}$$

We will assume that the solution V of the stationary problem admits of a double expansion in the eigenfunctions y_k of the spectral problem (3.11):

$$V = \sum_k C_k y_k, \quad i\omega V = \sum_k C_k \lambda_k y_k \tag{3.12}$$

A rigorous proof of this assumption is available only for a homogeneous cylinder and $R_3 = R_2$. In this case, in accordance with the terminology and the results of [10], problem (3.11) is strongly regular and the system of eigenfunctions and associated functions of problem (2.12) linearizing problem (3.11) by the spectral parameter is a Riesz basis in the space $W_2^1(R_1, R_3) \times L_2(R_1, R_3)$.

Then, by Proposition 2.2, the required coefficients in expansion (3.12) can be obtained from the formula

$$C_k = [(i\omega - \lambda_k)(y_k, y_k)_1]^{-1} \int_{R_1}^{R_3} f y_k r\, dr$$

where the scalar square is computed allowing for axial symmetry and using the first relationship in (2.10)

$$(y_k, y_k)_1 = \int_{R_1}^{R_3} 2\rho\lambda_k y_k^2 r dr + b y_k^2(R_3) R_3$$

Similarly using the solution of the spectral problem, we can construct the solution of the non-stationary problem, as in Sec. 1.

We wish to acknowledge the useful help and comments of A. A. Shkalikov.

REFERENCES

1. LYSMER J. and KUHLEMEYER R. L., Finite dynamic model for infinite media. *J. Engng. Mech. Div., Proc. ASCE* **95,** 4, 859–877, 1969.
2. MIRSAIDOV M., Solution of Lamb's problem by the finite-element method using radiation conditions. In *The Mechanics of Deformable Solids*, pp. 126–131. TGU, Tomsk, 1987.
3. PASHKOV I. A., On the orthogonality of the oscillation eigenmodes of a viscoelastic body. *Izv. Akad. Nauk SSSR, MTT* **4,** 104–111, 1989.
4. COLLATZ L., *Eigenvalue Problems*. Nauka, Moscow, 1968.
5. KELDYSH M. V., On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators. *Usp. Mat. Nauk* **26,** 4, 15–41, 1971.
6. SHKALIKOV A. A., Elliptical equations in a Hilbert space. Spectral problems associated with them. *Trudy Seminara im. I. G. Petrovskogo* **14,** 140–224, 1989.
7. MOSKVITIN V. V., *The Resistance of Viscoelastic Materials as it Applies to the Solid Fuel Charge of Rocket Engines*. Fizmatgiz, Moscow, 1972.
8. MIKHLIN S. G., *Variational Methods in Mathematical Physics*. Nauka, Moscow, 1970.
9. TIKHONOV A. N. and SAMARSKII A. A., *The Equations of Mathematical Physics*. Fizmatgiz, Moscow, 1972.
10. SHKALIKOV A. A., Boundary-value problems for ordinary differential equations with a parameter in the boundary conditions. *Trudy Seminara im. I. G. Petrovskogo* **9,** 160–229, 1983.

Translated by Z.L.